Teacher notes
Topic A

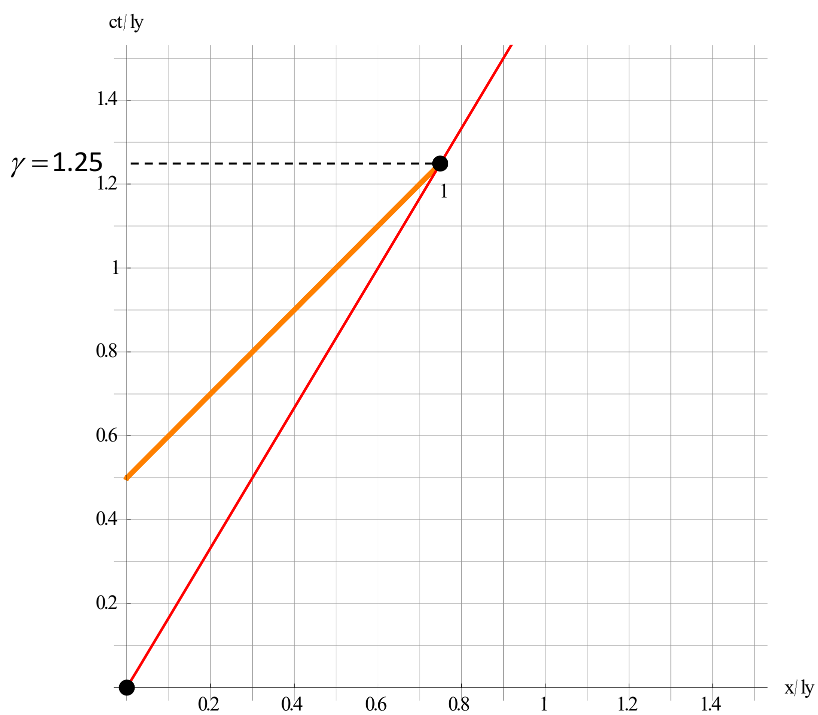
The birthday problem – a neat application of spacetime diagrams

In the textbook, an end of chapter problem asks:

A spacecraft leaves Earth moving at speed 0.60*c relative to Earth. Mandy, a spacecraft passenger, has her birthday that day. In a year's time (according to spacecraft clocks) she will have her birthday again. Her friends on Earth want to celebrate by sending her a light signal. When, by Earth clocks, should the light signal be sent so that it arrives at the spacecraft on Mandy's birthday?*

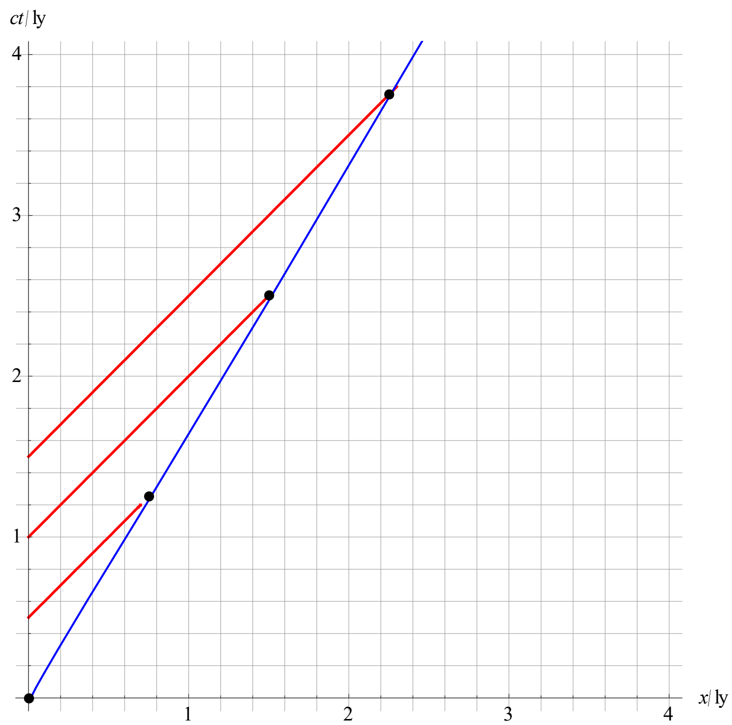
In the solutions, we solved the problem algebraically and it was rather involved. But here is a really easy solution based on spacetime diagrams. The graph shows the worldline of the spacecraft on the spacetime axes of the Earth.

The dots represent Mandy's birthdays (as measured by spacecraft clocks). All we have to do is draw a 45° line passing though the second birthday and see where it intersects the Earth time axis.

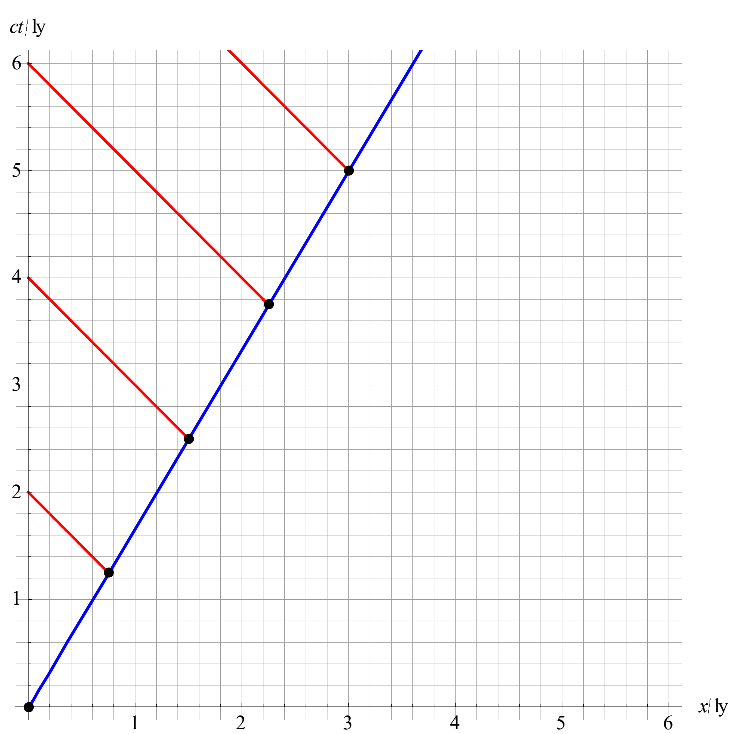


It does so at ct = 0.5 ly so the signal has to leave Earth 0.5 years after Mandy's first birthday at t = 0. This is the solution we got before but this is so much easier.

The next diagram shows how signals emitted every 6 months on earth arrive one year apart by Mandy's clocks.



Mandy politely responds with a light signal of her own. She emits the signals every year by her clocks. We see that the signals arrive at earth every two years by earth clocks. Time dilation is symmetric!



An extension

In the algebraic solution to the problem, we found that the time, *T*, between sending signals to the spacecraft and the time of arrivals of these signals at the spacecraft, τ, are related by

$$T=\tau\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,.$$

But we can imagine that the signals are actually wavefronts of a light wave sent to the spacecraft. In that case *T* is the period of the wave at the source and τ the period at the receiver (which is moving away from the source). This means that the frequencies of the wave at the source (f_S) and the receiver (f_R) are

$$\frac{1}{f_S}=\frac{1}{f_R}\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

or

$$f_R=f_S\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

which is the relativistic Doppler effect formula. It says that when the receiver (R) moves away from the source (S) the frequency received is less than that emitted.

In the case when $v<<c$,

$$\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\approx(1-\frac{1}{2}\frac{v}{c})(1-\frac{1}{2}\frac{v}{c})\approx 1-\frac{v}{c}$$

so that

$$f_R\approx f_S(1-\frac{v}{c})$$

$$f_R-f_S\approx -f_S\frac{v}{c}$$

$$\frac{\Delta f}{f_S}\approx -\frac{v}{c}$$

which is the approximate non-relativistic formula we learned in Topic C. The minus sign tells us that the observed frequency is less than the emitted frequency. If the source approaches the observer, then

$$f_R = f_S\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

and

$$\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \approx (1+\frac{1}{2}\frac{v}{c})(1+\frac{1}{2}\frac{v}{c}) \approx 1+\frac{v}{c}$$

leading to

$$f_R \approx f_S(1+\frac{v}{c})$$

$$f_R - f_S \approx f_S\frac{v}{c}$$

$$\frac{\Delta f}{f_S} \approx \frac{v}{c}$$